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Topological pressure of free semigroup actions under a mistake function for non-compact sets*

GUO Kai，*XIAO Qian*，*MA Dongkui*

School of Mathematics，South China University of Technology，Guangzhou 510641，China

Abstract：Topological pressure is a core concept of dynamic systems and ergodic theory，which plays an important role in the study of thermodynamic formalism. As the physical process evolves，it is natural for the evolution process to produce changes or some errors in the orbit calculation. However，a self-adaptable system should decrease errors over time. In this paper，we introduce two definitions of topological pressure of free semigroup actions under a mistake function by using C-P structure and prove that they are equivalent. Furthermore，we show that the topological pressure of free semigroup actions under a mistake function on a non-compact subset is equivalent to the topological pressure of free semigroup actions of the subset without mistake function. As an application，we use the above theorem to show that the topological pressure of free semigroup actions defined by mean metric is equivalent to the topological pressure of free semigroup actions defined by Bowen metric.

Key words：free semigroup actions；topological pressure；mistake function；mean metric；C-P structure

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非紧集上错误函数下自由半群作用的拓扑压

郭锴，肖倩，马东魁

华南理工大学数学学院，广东 广州 510641

摘　要：拓扑压是动力系统和遍历理论中的核心概念，它在热力学形式的研究中具有重要作用。随着物理过程的演化，演化过程在轨道计算中产生变化或误差是很自然的。然而，一个自适应系统应该随着时间的推移减少误差。本文利用C-P结构给出了错误函数下自由半群作用的拓扑压的两个定义并证明它们是等价的，此外，还证明了非紧集上错误函数下自由半群作用的拓扑压和没有错误函数下自由半群作用的拓扑压是等价的。最后，应用上述定理举例证明了平均度量下的自由半群作用的拓扑压等价于Bowen度量下的自由半群作用的拓扑压。

关键词：自由半群作用；拓扑压；错误函数；平均度量；C-P结构

Topological pressure，introduced by Ruelle[1] and Walters[2]，is a core concept in dynamical system and ergodic theory，and plays an important role in the study of thermodynamic formalism. From the viewpoint of dimension theory，Pesin and Pitskel[3-4] studied the topological pressure of non-compact subsets by using carátheory structure（C-P structure），and proposed the topological pressure of non-compact sets in dynamical systems，

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作者简介：郭锴（1997年生），女；**研究方向**：动力系统；E-mail：1610411248@qq. com
通信作者：马东魁（1964年生），男；**研究方向**：动力系统；E-mail：dkma@scut. edu. cn

which is the generalization of Bowen's[5] topological entropy defined by non-compact sets. As the physical process evolves, it is natural for the evolution process to produce changes or some errors in the orbit calculation. However, a self-adaptable system should decrease errors over time. This prompted Cheng, Zhao et al[6] to study the dynamical systems under a mistake function. They defined the pressure for asymptotically sub-additive potentials under a mistake function, and proved that the topological pressure under a mistake function is equivalent to the topological pressure without mistake function by using the ergodic theory. Later, Chen et al[7] gave the concepts of topological pressure with mistake function and showed that the topological pressure under a mistake function on any subset is the same as the classical Pesin pressure of the subset in dynamical systems. This means that the topological pressure under a mistake function in the dynamical system is adaptive, which generalizes the result in the additive case in [3].

With the development of research on dynamical systems, the dynamical systems of group action has attracted people's attention. To study some questions, Xiao et al[8] gave the definitions of topological pressure and upper and lower capacity topological pressures of a free semigroup action by using C-P structure, and obtained some properties of them. Naturally, we wonder if the topological pressure of free semigroup actions has the similar result in [7]. In order to answer this question, in this paper we introduce the definitions of topological pressure of free semigroup actions under a mistake function and show the topological pressure of free semigroup actions under a mistake function is the same as the topological pressure of free semigroup actions defined by Xiao and Ma[8].

Furthermore, Gröger et al[9] showed that the entropy of the whole system with Bowen metric is equal to the entropy with mean metric. As an application, we prove that the topological pressure of free semigroup actions defined by Bowen metric coincides with the topological pressure of free semigroup actions defined by mean metric on a non-compact subset.

This paper is organized as follows. In section 1, we give some preliminaries. In section 2, we introduce two definitions of topological pressure of free semigroup actions under a mistake function and prove the main results. Finally, we give an application.

1　Preliminaries

1.1　Words and sequences

Let F_m^+ denote the set of all finite words of symbols $0, 1, \cdots, m-1$. For any $w \in F_m^+$, the length of w, denoted by $|w|$, is defined the digits of symbols in w. Obviously, F_m^+ with respect to the law of composition is a free semigroup with m generators. We write $w' \leqslant w$ if there exists a word $w'' \in F_m^+$ such that $w = w''w'$. For $w = i_1 \cdots i_k \in F_m^+$, denote $\bar{w} = i_k \cdots i_1$.

Denote by Σ_m the set of all two-side infinite sequences of symbols $0, 1, \cdots, m-1$, that is,

$$\Sigma_m = \{ \omega = (\cdots, i_{-1}, i_0, i_1, \cdots) : i_j = 0, 1, \cdots, m-1 \text{ for all integer } j \}.$$

Suppose that $\omega \in \Sigma_m, w \in F_m^+, a, b$ are integers, and $a \leqslant b$. We write $\omega|_{[a,b]} = w$ if $w = i_a i_{a+1} \cdots i_{b-1} i_b$.

Denote by Σ_m^+ the set of all one-side infinite sequences of symbols $0, 1, \cdots, m-1$:

$$\Sigma_m^+ = \{ \omega = (i_0, i_1, \cdots) : i_j = 0, 1, \cdots, m-1 \text{ for all integer } j \}.$$

1.2　Mistake function

Let's recall the definition of the mistake function in [7], which is a little bit different from [6,10].

Definition 1　Given $\varepsilon_0 > 0$, the function $g: \mathbb{N} \times (0, \varepsilon_0] \to \mathbb{R}$ is called a mistake function if $g(n, \varepsilon) \leqslant g(n+1, \varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$ and $n \in \mathbb{N}$ and

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{g(n, \varepsilon)}{n} = 0.$$

For a mistake function g, if $\varepsilon > \varepsilon_0$, set $g(n, \varepsilon) = g(n, \varepsilon_0)$.

1.3 Definition of topological pressure of free semigroup actions by using open covers

Let (X,d) be a compact metric space and G be the free semigroup generated by $f_0,f_1,\cdots,f_{m-1}$, where $f_i(0\leqslant i\leqslant m-1)$ is continuous transformations from X to itself. Given $\varphi_0,\varphi_1,\cdots,\varphi_{m-1}\in C(X,\mathbb{R})$, denote $\Phi=\{\varphi_0,\varphi_1,\cdots,\varphi_{m-1}\}$. For simplicity of notation, we write f_w instead of $f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_n}$, where $w=i_1i_2\cdots i_n\in F_m^+$. Obviously, $f_{ww'}=f_wf_{w'}$ for any $w,w'\in F_m^+$. For $w=i_1i_2\cdots i_n\in F_m^+$, denote

$$S_w\Phi(x):=\varphi_{i_1}(x)+\varphi_{i_2}(f_{i_1}(x))+\cdots+\varphi_{i_n}(f_{i_{n-1}i_{n-2}\cdots i_1}(x)).$$

Xiao and Ma[8] introduced the notion of topological pressure of free semigroup actions by C-P structure as follows:

Considering a finite open cover $\mathcal{U}$ of X, write $|\mathcal{U}|=\max\{|U|:U\in\mathcal{U}\}$, and let

$$\mathcal{S}_{n+1}(\mathcal{U}):=\{\mathbf{U}=(U_0,U_1,\cdots,U_n):\ \mathbf{U}\in\mathcal{U}^{n+1}\},$$

where $\mathcal{U}^{n+1}=\prod_{i=1}^{n+1}\mathcal{U}$ and $n\geqslant 0$. For any string $\mathbf{U}\in\mathcal{S}_{n+1}(\mathcal{U})$, define the length of $\mathbf{U}$ as $m(\mathbf{U}):=n+1$. We put $\mathcal{S}=\mathcal{S}(\mathcal{U})=\bigcup_{n\geqslant 1}\mathcal{S}_n(\mathcal{U})$. For any $\omega=(i_1,i_2,\cdots,i_n,\cdots)\in\Sigma_m^+$, $n\geqslant 1$, and a given string $\mathbf{U}=(U_0,U_1,\cdots,U_n)\in\mathcal{S}_{n+1}(\mathcal{U})$, we associate the set

$$X_\omega(\mathbf{U})=\{x\in X:\ x\in U_0,\ f_{i_j}\circ\cdots\circ f_{i_1}(x)\in U_j,\ j=1,2,\cdots,n\}.$$

If $w_{\mathbf{U}}=\omega|_{[0,n-1]}=i_1i_2\cdots i_n\in F_m^+$, we also denote $X_\omega(\mathbf{U})$ by $X_{w_{\mathbf{U}}}(\mathbf{U})$ for brevity.

Given $w\in F_m^+$, $|w|=N$, $Z\subset X$ and $\alpha\in\mathbb{R}$, define

$$M_w(Z,\alpha,\Phi,\mathcal{U},N):=\inf_{\mathcal{G}_w}\left\{\sum_{\mathbf{U}\in\mathcal{G}_w}\exp\left(-\alpha m(\mathbf{U})+\sup_{x\in X_{w_{\mathbf{U}}}(\mathbf{U})}S_{w_{\mathbf{U}}}\Phi(x)\right)\right\},$$

where the infimum is taken over all finite or countable collections of strings $\mathcal{G}_w\subset\mathcal{S}(\mathcal{U})$ such that $m(\mathbf{U})\geqslant N+1$ for all $\mathbf{U}\in\mathcal{G}_w$ and $\mathcal{G}_w$ covers Z (i. e. for any $\mathbf{U}\in\mathcal{G}_w$, there is $w_{\mathbf{U}}\in F_m^+$ such that $\overline{w}\leqslant\overline{w_{\mathbf{U}}}$ and $\bigcup_{\mathbf{U}\in\mathcal{G}_w}X_{w_{\mathbf{U}}}(\mathbf{U})\supset Z$).

Let

$$M(Z,\alpha,\Phi,\mathcal{U},N)=\frac{1}{m^N}\sum_{|w|=N}M_w(Z,\alpha,\Phi,\mathcal{U},N).$$

We can easily verify that the function $M(Z,\alpha,\Phi,\mathcal{U},N)$ is non-decreasing as N increases. Therefore, the following limit exists

$$m(Z,\alpha,\Phi,\mathcal{U})=\lim_{N\to\infty}M(Z,\alpha,\Phi,\mathcal{U},N).$$

There is a critical value of α at which $m(Z,\alpha,\Phi,\mathcal{U})$ jumps from ∞ to 0. Denote

$$P_Z(G,\Phi,\mathcal{U})=\inf\{\alpha:\ m(Z,\alpha,\Phi,\mathcal{U})=0\}=\sup\{\alpha:\ m(Z,\alpha,\Phi,\mathcal{U})=\infty\}.$$

The topological pressure of a free semigroup G with respect to Φ on the set Z is

$$P_Z(G,\Phi)=\lim_{|\mathcal{U}|\to 0}P_Z(G,\Phi,\mathcal{U}).$$

1.4 Definition of topological pressure of free semigroup actions by using the center of Bowen ball

First, recall the definitions of the Bowen metric and (w,δ)-Bowen ball.

Let X be a compact metric space with metric d, $f_0,f_1,\cdots,f_{m-1}$ continuous transformations from X to itself. Suppose that a free semigroup G with m generators $f_0,f_1,\cdots,f_{m-1}$ acts on X. Denote $\Phi=\{\varphi_0,\varphi_1,\cdots,\varphi_{m-1}\}$, where $\varphi_0,\varphi_1,\cdots,\varphi_{m-1}\in C(X,\mathbb{R})$. For each $w\in F_m^+$, a new metric d_w on X (named Bowen metric) is given by

$$d_w(x_1,x_2)=\max_{w'\leqslant\overline{w}}d(f_{w'}(x_1),f_{w'}(x_2)).$$

Clearly, if $\overline{w'}\leqslant\overline{w''}$, then $d_{w'}(x_1,x_2)\leqslant d_{w''}(x_1,x_2)$ for all $x_1,x_2\in X$.

Fix a number $\delta>0$. Given $w\in F_m^+$ and a point $x\in X$, define the (w,δ)-Bowen ball at x by

$$B_w(x,\delta)=\{y\in X:\ d_w(x,y)<\delta\}.$$

Based on the work of Climenhaga in [11], Xiao and Ma[8] defined the topological pressure of free semigroup actions by using the center of Bowen ball. Now, let us recall the definition of topological pressure of free semigroup actions by the center of Bowen ball in [8].

Let $\mathcal{F}$ denote the collection of Bowen ball, that is,

$$\mathcal{F} = \left\{ B_w(x,\delta): x \in X, w \in F_m^+ \right\}.$$

Given $w \in F_m^+$, $|w| = N \in \mathbb{N}$, $Z \subset X$ and $\alpha \in \mathbb{R}$, define

$$M'_w(Z,\alpha,\Phi,\delta,N) := \inf_{\mathcal{G}_w} \left\{ \sum_{B_{w'}(x,\delta) \in \mathcal{G}_w} \exp\left(-\alpha \cdot |w'| + S_{w'}\Phi(x)\right) \right\},$$

where the infimum is taken over all finite or countable collections $\mathcal{G}_w \subset \mathcal{F}$ covering Z (i. e., for any $B_{w'}(x,\delta) \in \mathcal{G}_w$, $\bar{w} \leqslant \overline{w'}$ and $\bigcup_{B_{w'}(x,\delta)\in\mathcal{G}_w} B_{w'}(x,\delta) \supset Z$).

Set

$$M'(Z,\alpha,\Phi,\delta,N) = \frac{1}{m^N} \sum_{|w|=N} M'_w(Z,\alpha,\Phi,\delta,N).$$

It is easy to verify that the function $M'(Z,\alpha,\Phi,\delta,N)$ is non-decreasing as N increases. Therefore, there exists the limit

$$m'(Z,\alpha,\Phi,\delta) = \lim_{N\to\infty} M'(Z,\alpha,\Phi,\delta,N).$$

Same as above, denote the critical value of α by

$$P'_Z(G,\Phi,\delta) = \inf\{\alpha: m'(Z,\alpha,\Phi,\delta) = 0\} = \sup\{\alpha: m'(Z,\alpha,\Phi,\delta) = \infty\}.$$

The topological pressure of a free semigroup G with respect to Φ on the set Z is

$$P_Z(G,\Phi) = \lim_{\delta\to 0} P'_Z(G,\Phi,\delta).$$

2 Main results and example

Based on [6-7], in this section, we first give two concepts of topological pressure of free semigroup actions under a mistake function by using open covers and the center of mistake Bowen ball respectively. In addition, we show that these two concepts are equivalent to the definitions of the topological pressure of free semigroup actions introduced by Xiao and Ma[8]. Finally, we give a notion of mean metric and show that the topological pressure of free semigroup actions under a mistake function defined by Bowen metric agrees with the topological pressure of free semigroup actions defined by mean metric.

Let $f_0, f_1, \cdots, f_{m-1}$ be the continuous transformations from compact metric space (X,d) to itself. Denote G the free semigroup with m generators $f_0, f_1, \cdots, f_{m-1}$ acting on X. Given $\varphi_0, \varphi_1, \cdots, \varphi_{m-1} \in C(X,\mathbb{R})$, denote $\Phi = \{\varphi_0, \varphi_1, \cdots, \varphi_{m-1}\}$.

2.1 Definition of topological pressure of free semigroup actions under a mistake function by using open covers

Considering a finite open cover $\mathcal{U}$ of X. For any string $\mathbf{U} \in \mathcal{S}_{n+1}(\mathcal{U})$, let

$$\mathbf{U}^g = \left\{ \mathbf{U}^* = (U_0^*, U_1^*, \cdots, U_n^*): \sharp\{U_j^* \neq U_j, j = 0,1,\cdots,n\} \leqslant g\left(m(\mathbf{U}), |\mathcal{U}|\right) \right\},$$

define the length of $\mathbf{U}^g$ as $m(\mathbf{U}^g) = m(\mathbf{U}) := n+1$, and let $\mathcal{S}_{n+1}^g(\mathcal{U})$ denote the set of such $\mathbf{U}^g$ of length $m(\mathbf{U}^g) = n+1$. We put $\mathcal{S}^g(\mathcal{U}) = \bigcup_{n\geqslant 1} \mathcal{S}_n^g(\mathcal{U})$. For any $\omega = (i_1, i_2, \cdots, i_n, \cdots) \in \Sigma_m^+$, $n \geqslant 1$, and given $\mathbf{U}^g \in \mathcal{S}_{n+1}^g(\mathcal{U})$, we associate the set

$$X_\omega(\mathbf{U}^g) = \left\{ x \in X: \exists\, \mathbf{U}^* \in \mathbf{U}^g \text{ such that } x \in U_0^*, f_{i_j} \circ \cdots \circ f_{i_1}(x) \in U_j^*, j = 1,\cdots,n \right\}.$$

If $w_{\mathbf{U}^g} = \omega|_{[0,n-1]} = i_1 i_2 \cdots i_n \in F_m^+$, we also denote $X_\omega(\mathbf{U}^g)$ by $X_{w_{\mathbf{U}^g}}(\mathbf{U}^g)$ for the sake of convenience.

Given $w \in F_m^+$, $|w| = N$, $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M_w(Z,\alpha,\Phi,\mathcal{U},N,g) := \inf_{\mathcal{G}_w^g}\left\{\sum_{\mathbf{U}^g\in\mathcal{G}_w^g}\exp\left(-\alpha m(\mathbf{U}) + \sup_{x\in X_{w_{\mathbf{U}^g}}(\mathbf{U}^g)} S_{w_{\mathbf{U}^g}}\Phi(x)\right)\right\},$$

where the infimum is taken over all finite or countable collections of strings $\mathcal{G}_w^g \subset \mathcal{S}^g(\mathcal{U})$ such that $m(\mathbf{U}^g) \geqslant N+1$ for all $\mathbf{U}^g \in \mathcal{G}_w^g$ and $\mathcal{G}_w^g$ covers Z (i. e. for any $\mathbf{U}^g \in \mathcal{G}_w^g$, there is $w_{\mathbf{U}^g} \in F_m^+$ such that $\bar{w} \leqslant \overline{w_{\mathbf{U}^g}}$ and $\bigcup_{\mathbf{U}^g\in\mathcal{G}_w^g} X_{w_{\mathbf{U}^g}}(\mathbf{U}^g) \supset Z$).

Let

$$M(Z,\alpha,\Phi,\mathcal{U},N,g) = \frac{1}{m^N}\sum_{|w|=N} M_w(Z,\alpha,\Phi,\mathcal{U},N,g).$$

We can easily verify that the function $M(Z,\alpha,\Phi,\mathcal{U},N,g)$ is non-decreasing as N increases. This guarantees the existence of the following limit

$$m(Z,\alpha,\Phi,\mathcal{U},g) := \lim_{N\to\infty} M(Z,\alpha,\Phi,\mathcal{U},N,g).$$

There exists a critical value of the parameter α, which we will denote by $P_Z(G,\Phi,\mathcal{U},g)$, where $m(Z,\alpha,\Phi,\mathcal{U},g)$ jumps from ∞ to 0, that is,

$$m(Z,\alpha,\Phi,\mathcal{U},g) = \begin{cases} 0, & \alpha > P_Z(G,\Phi,\mathcal{U},g), \\ \infty, & \alpha < P_Z(G,\Phi,\mathcal{U},g). \end{cases}$$

The following theorem discusses the connection between the above definition and the topological pressure of free semigroup actions defined by using open covers.

Theorem 1 For any set $Z \subset X$, the following limit exists and equals to the topological pressure of a free semigroup action G, that is,

$$P_Z(G,\Phi) = P_Z(G,\Phi,g) =: \lim_{|\mathcal{U}|\to 0} P_Z(G,\Phi,\mathcal{U},g).$$

Proof We use the analogous method as that of [3]. On the one hand, given $w = i_1 i_2 \cdots i_N \in F_m^+$, for any $\mathbf{U} \in \mathcal{S}_{n+1}(\mathcal{U})$, we denote the word that corresponds to $\mathbf{U}$ by $w_{\mathbf{U}}$ such that $\bar{w} \leqslant \overline{w_{\mathbf{U}}}$, and corresponds to $\mathbf{U}^g$ by $w_{\mathbf{U}^g}$ such that $w_{\mathbf{U}^g} = w_{\mathbf{U}}$, then we have

$$X_{w_{\mathbf{U}}}(\mathbf{U}) \subset X_{w_{\mathbf{U}}}(\mathbf{U}^g).$$

Let

$$\lambda = \lambda(\Phi) := \max\left\{\left\|\varphi_j\right\| : j = 0,1,\cdots,m-1\right\}, \tag{1}$$

and

$$\beta = \beta(\mathcal{U}) := \sup\left\{\left|\varphi_j(x) - \varphi_j(y)\right| : x,y \in U, U \in \mathcal{U}, j = 0,1,\cdots,m-1\right\}, \tag{2}$$

where $\left\|\varphi_j\right\| := \sup_{x\in X}\left|\varphi_j(x)\right|$. Because of $\varphi_j \in C(X,\mathbb{R})$, φ_j is uniformly continuous for $0 \leqslant j \leqslant m-1$. Then β is finite and $\lim_{|\mathcal{U}|\to 0}\beta = 0$. Furthermore, for any $x \in X_{w_{\mathbf{U}}}(\mathbf{U}^g)$ and $y \in X_{w_{\mathbf{U}}}(\mathbf{U})$, we have

$$\left|S_{w_{\mathbf{U}}}\Phi(x) - S_{w_{\mathbf{U}}}\Phi(y)\right| \leqslant 2\lambda g\big(m(\mathbf{U}), |\mathcal{U}|\big) + \beta m(\mathbf{U}).$$

Now, given $w \in F_m^+$ with $|w| = N$, then for any $\varepsilon > 0$, there is $\mathcal{G}_w^* \subset \mathcal{S}(\mathcal{U})$ covering Z such that

$$\begin{aligned} M_w(Z,\alpha,\Phi,\mathcal{U},N) &\geqslant \sum_{\mathbf{U}\in\mathcal{G}_w^*}\exp\left(-(\alpha+\varepsilon)m(\mathbf{U}) + \sup_{x\in X_{w_{\mathbf{U}}}(\mathbf{U})} S_{w_{\mathbf{U}}}\Phi(x)\right) \\ &\geqslant \sum_{\mathbf{U}^g\in\mathcal{G}_w'}\exp\left(-(\alpha+\varepsilon)m(\mathbf{U}) + \sup_{x\in X_{w_{\mathbf{U}}}(\mathbf{U}^g)} S_{w_{\mathbf{U}}}\Phi(x) - 2\lambda g(m(\mathbf{U}),|\mathcal{U}|) - \beta m(\mathbf{U})\right) \\ &\geqslant \sum_{\mathbf{U}^g\in\mathcal{G}_w'}\exp\left(-(\alpha+\varepsilon+\beta+\gamma)m(\mathbf{U}) + \sup_{x\in X_{w_{\mathbf{U}}}(\mathbf{U}^g)} S_{w_{\mathbf{U}}}\Phi(x)\right), \end{aligned}$$

where $\mathcal{G}'_w = \{\mathbf{U}^g : \mathbf{U} \in \mathcal{G}^*_w, w_{\mathbf{U}^g} = w_{\mathbf{U}}\}$，$\gamma$ is a function such that

$$2\lambda g\big(m(\mathbf{U}), |\mathcal{U}|\big) \leqslant \gamma m(\mathbf{U}) \tag{3}$$

for all $m(\mathbf{U}) \geqslant N + 1$ and $\gamma \to 0$ as $|\mathcal{U}| \to 0$. Moreover，we can get

$$M_w(Z, \alpha, \Phi, \mathcal{U}, N) \geqslant M_w(Z, \alpha + \varepsilon + \beta + \gamma, \Phi, \mathcal{U}, N, g),$$

which implies

$$M(Z, \alpha, \Phi, \mathcal{U}, N) \geqslant M(Z, \alpha + \varepsilon + \beta + \gamma, \Phi, \mathcal{U}, N, g).$$

Taking the limit $N \to \infty$ yields

$$m(Z, \alpha, \Phi, \mathcal{U}) \geqslant m(Z, \alpha + \varepsilon + \beta + \gamma, \Phi, \mathcal{U}, g).$$

Therefore,

$$P_Z(G, \Phi, \mathcal{U}) \geqslant P_Z(G, \Phi, \mathcal{U}, g) - \varepsilon - \beta - \gamma,$$

and as $|\mathcal{U}| \to 0$，that is，$\beta \to 0$，$\gamma \to 0$, we obtain

$$P_Z(G, \Phi) \geqslant \limsup_{|\mathcal{U}| \to 0} P_Z(G, \Phi, \mathcal{U}, g) - \varepsilon.$$

By the arbitrariness of ε，we have

$$P_Z(G, \Phi) \geqslant \limsup_{|\mathcal{U}| \to 0} P_Z(G, \Phi, \mathcal{U}, g).$$

On the other hand，note that $w_{\mathbf{U}} = w_{\mathbf{U}^g}$，for any $\mathbf{U}^g \in \mathcal{S}^g_m(\mathcal{U})$, we can find

$$M: = \sum_{i=0}^{g(m, |\mathcal{U}|)} \binom{m}{i} (\#\mathcal{U} - 1)^i$$

strings $\mathbf{U} \in \mathcal{S}_m(\mathcal{U})$ such that

$$X_{w_{\mathbf{U}}}(\mathbf{U}^g) \subset \bigcup_{\mathbf{U} \in \tau} X_{w_{\mathbf{U}}}(\mathbf{U}),$$

where τ is the collection of such $\mathbf{U}$. Using Stirling formula，there exists $\gamma_1 > 0$ such that

$$M \leqslant \exp\big(m(\mathbf{U})\gamma_1\big)$$

for all $m(\mathbf{U}) \geqslant N + 1$, and $\gamma_1 \to 0$ as $|\mathcal{U}| \to 0$. Given $w \in F^+_m$ with $|w| = N$，and $\varepsilon > 0$，there is $\mathcal{G}''_w \subset \mathcal{S}^g(\mathcal{U})$ covering Z such that

$$\begin{aligned}
M_w(Z, \alpha, \Phi, \mathcal{U}, N) &\geqslant \sum_{\mathbf{U}^g \in \mathcal{G}''_w} \exp\left(-(\alpha + \varepsilon)m(\mathbf{U}) + \sup_{x \in X_{w_{\mathbf{U}^g}}(\mathbf{U}^g)} S_{w_{\mathbf{U}^g}}\Phi(x) \right) \\
&\geqslant \frac{1}{M} \sum_{\mathbf{U} \in \mathcal{G}'''_w} \exp\left(-(\alpha + \varepsilon)m(\mathbf{U}) + \sup_{x \in X_{w_{\mathbf{U}}}(\mathbf{U})} S_{w_{\mathbf{U}}}\Phi(x) - 2\lambda g(m(\mathbf{U}), |\mathcal{U}|) - \beta m(\mathbf{U}) \right) \\
&\geqslant \sum_{\mathbf{U} \in \mathcal{G}'''_w} \exp(-(\alpha + \varepsilon + \beta + \gamma + \gamma_1)m(\mathbf{U}) + \sup_{x \in X_{w_{\mathbf{U}}}(\mathbf{U})} S_{w_{\mathbf{U}}}\Phi(x)),
\end{aligned}$$

where $\mathcal{G}'''_w = \{\mathbf{U} : \mathbf{U}^g \in \mathcal{G}''_w, w_{\mathbf{U}^g} = w_{\mathbf{U}}\}$ denotes the set of all elements of all τ corresponding to each $\mathbf{U}^g \in \mathcal{G}''_w$，$\beta$，$\lambda$ are given by (1)，(2) respectively and γ satisfies inequality (3). Moreover，we can get

$$M_w(Z, \alpha, \Phi, \mathcal{U}, N, g) \geqslant M_w(Z, \alpha + \varepsilon + \beta + \gamma + \gamma_1, \Phi, \mathcal{U}, N),$$

which implies

$$M(Z, \alpha, \Phi, \mathcal{U}, N, g) \geqslant M(Z, \alpha + \varepsilon + \beta + \gamma + \gamma_1, \Phi, \mathcal{U}, N).$$

Take the limit $N \to \infty$ and we get

$$m(Z, \alpha, \Phi, \mathcal{U}, g) \geqslant m(Z, \alpha + \varepsilon + \beta + \gamma + \gamma_1, \Phi, \mathcal{U}).$$

Therefore,

$$P_Z(G, \Phi, \mathcal{U}) \leqslant P_Z(G, \Phi, \mathcal{U}, g) + \varepsilon + \beta + \gamma + \gamma_1,$$

and as $|\mathcal{U}| \to 0$，that is，$\beta \to 0$，$\gamma \to 0$，$\gamma_1 \to 0$，we obtain

$$P_Z(G, \Phi) \leqslant \liminf_{|\mathcal{U}| \to 0} P_Z(G, \Phi, \mathcal{U}, g) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary，we have

$$P_Z(G,\Phi) \leqslant \liminf_{|\mathcal{U}|\to 0} P_Z(G,\Phi,\mathcal{U},g).$$

This completes the proof.

The quantity $P_Z(G,\Phi,g)$ is called the topological pressure of a free semigroup G under a mistake function g with respect to Φ on the set Z.

2.2 Definition of topological pressure of free semigroup actions under a mistake function by using the center of mistake Bowen balls

Now, we introduce a definition of mistake Bowen ball $B_w(g;x,\delta)$ for a given $w \in F_m^+$.

Fix a number $\delta > 0$. Given $w \in F_m^+$ and a point $x \in X$, the mistake Bowen ball $B_w(g;x,\delta)$ centered at x with radius δ and length $|w|+1$ associated to the mistake function g is given by the following set, i. e.,

$$B_w(g;x,\delta) := \left\{ y \in X: \exists \mathcal{A}'_w \subset \mathcal{A}_w, \sharp(\mathcal{A}_w \backslash \mathcal{A}'_w) \leqslant g(|w|+1,\delta), \max_{w'' \in \mathcal{A}'_w}\{d(f_{w''}(x), f_{w''}(y))\} \leqslant \delta \right\},$$

where $\mathcal{A}_w = \{w': w' \leqslant \bar{w}\}$.

It is obvious that the (w,δ)-Bowen ball is a subset of $B_w(g;x,\delta)$.

Now, we describe another approach to redifine the topological pressure of a free semigroup G under a mistake function g by the center of mistake Bowen ball.

Define the collection of subsets

$$\mathcal{F}^g = \{B_w(g;x,\delta): x \in X, w \in F_m^+\}.$$

Given $w \in F_m^+$ with $|w| = N$, $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M'_w(Z,\alpha,\Phi,\delta,N,g) := \inf_{\mathcal{G}_w^g} \left\{ \sum_{B_{w'}(g;x,\delta) \in \mathcal{G}_w^g} \exp\left(-\alpha \cdot |w'| + S_{w'}\Phi(x)\right) \right\},$$

where the infimum is taken over all finite or countable collections $\mathcal{G}_w^g \subset \mathcal{F}^g$ covering Z (i. e., for any $B_{w'}(g;x,\delta) \in \mathcal{G}_w^g$, $\bar{w} \leqslant \overline{w'}$ and $\bigcup_{B_{w'}(g;x,\delta)\in \mathcal{G}_w^g} B_{w'}(g;x,\delta) \supset Z$).

Put

$$M'(Z,\alpha,\Phi,\delta,N,g) = \frac{1}{m^N} \sum_{|w|=N} M'_w(Z,\alpha,\Phi,\delta,N,g).$$

It is easy to verify that the function $M'(Z,\alpha,\Phi,\delta,N,g)$ is non-decreasing as N increases. This guarantees the existence of the limit

$$m'(Z,\alpha,\Phi,\delta,g) = \lim_{N\to\infty} M'(Z,\alpha,\Phi,\delta,N,g).$$

There exists a critical value of the parameter α, which we will denote by $P'_Z(G,\Phi,\delta,g)$, where $m'(Z,\alpha,\Phi,\delta,g)$ jumps from ∞ to 0, namely,

$$m'(Z,\alpha,\Phi,\delta,g) = \begin{cases} 0, & \alpha > P'_Z(G,\Phi,\delta,g), \\ \infty, & \alpha < P'_Z(G,\Phi,\delta,g). \end{cases}$$

The following theorem proves that the two definitions of topological pressure of free semigroup actions under a mistake function are equivalent.

Theorem 2 For any set $Z \subset X$, the following limit exists and equals to $P_Z(G,\Phi,g)$ defined by using open covers, that is,

$$P_Z(G,\Phi,g) = \lim_{\delta\to 0} P'_Z(G,\Phi,\delta,g).$$

Proof Our proof is adapted from Climenhaga's elegament argument in [11]. On the one hand, given $\delta > 0$, let

$$\lambda = \lambda(\Phi) := \max\left\{ \|\varphi_j\|: j = 0,1,\cdots,m-1 \right\},$$

and

$$\varepsilon(\delta) := \sup\left\{ |\varphi_i(x) - \varphi_i(y)|: d(x,y) < \delta, i = 0,1,\cdots,m-1 \right\},$$

where $\|\varphi_j\| := \sup_{x\in X}|\varphi_j(x)|$. Observe that since $\varphi_i \in C(X,\mathbb{R})$ and X is compact, φ_i is in fact uniformly continuous, hence $\varepsilon(\delta)$ is finite and $\lim_{\delta\to 0}\varepsilon(\delta)=0$. Now fix $\delta>0$, choose a finite open cover $\mathcal{U}$ of X with $|\mathcal{U}|<\delta$ and let $\mathcal{L}(\mathcal{U})$ be the Lebesgue number of $\mathcal{U}$. Given $x\in X$, $w\in F_m^+$ and $|w|=N$, let $\mathcal{G}_w^g=\left\{B_{w'}\left(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right):\ x\in X, \bar{w}\leqslant \overline{w'}\right\}$ be a open cover of Z, then for each $B_{w'}\left(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)\in\mathcal{G}_w^g$, there exists $\mathbf{U}^{g'}\in\mathcal{S}_{|w'|+1}^{g'}(\mathcal{U})$ such that

$$B_{w'}\left(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)\subset X_{w'}(\mathbf{U}^{g'})$$

where $g'=g\left(|w'|+1,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)$. Then for any $y\in X_{w'}(\mathbf{U}^{g'})$, we have

$$\left|S_{w'}\Phi(y)-S_{w'}\Phi(x)\right|\leqslant 2\lambda g'\left(m(\mathbf{U}^{g'}),\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)+m(\mathbf{U}^{g'})\varepsilon(\delta).$$

Set

$$\mathcal{G}_w'=\left\{\mathbf{U}^{g'}:\ B_{w'}\left(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)\subset X_{w'}(\mathbf{U}^{g'})\right\}.$$

Note that $m(\mathbf{U})=m(\mathbf{U}^{g'})=|w'|+1$, then

$$\begin{aligned}
M_w(Z,\alpha,\Phi,\mathcal{U},N,g') &\leqslant \sum_{\mathbf{U}^{g'}\in\mathcal{G}_w'}\exp\left(-\alpha m(\mathbf{U})+\sup_{y\in X_{w'}(\mathbf{U}^{g'})}S_{w'}\Phi(y)\right)\\
&\leqslant \sum_{B_{w'}(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U}))\in\mathcal{G}_w^g}\exp\left(-\alpha m(\mathbf{U})+S_{w'}\Phi(x)+2\lambda g'(m(\mathbf{U}),\ \frac{1}{2}\mathcal{L}(\mathcal{U}))+m(\mathbf{U})\varepsilon(\delta)\right)\\
&\leqslant e^{\varepsilon(\delta)-\alpha}\cdot\sum_{B_{w'}(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U}))\in\mathcal{G}_w^g}\exp\left(-|w'|(\alpha-\varepsilon(\delta))+2\lambda g'(m(\mathbf{U}),\ \frac{1}{2}\mathcal{L}(\mathcal{U}))+S_{w'}\Phi(x)\right)\\
&\leqslant e^{\varepsilon(\delta)+\gamma-\alpha}\cdot\sum_{B_{w'}(g;\ x,\ \frac{1}{2}\mathcal{L}(\mathcal{U}))\in\mathcal{G}_w^g}\exp\left(-|w'|(\alpha-\varepsilon(\delta)-\gamma)+S_{w'}\Phi(x)\right),
\end{aligned}$$

where γ is a function such that $2\lambda g'\left(m(\mathbf{U}),\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right)\leqslant\gamma m(\mathbf{U})$ for all $m(\mathbf{U})\geqslant N+1$ and $\gamma\to 0$ as $|\mathcal{U}|\to 0$. Moreover, we can get

$$M_w(Z,\alpha,\Phi,\mathcal{U},N,g')\leqslant e^{\varepsilon(\delta)+\gamma-\alpha}\cdot M_w'\left(Z,\alpha-\varepsilon(\delta)-\gamma,\Phi,\ \frac{1}{2}\mathcal{L}(\mathcal{U}),N,g\right),$$

which implies

$$M(Z,\alpha,\Phi,\mathcal{U},N,g')\leqslant e^{\varepsilon(\delta)+\gamma-\alpha}\cdot M'\left(Z,\alpha-\varepsilon(\delta)-\gamma,\Phi,\ \frac{1}{2}\mathcal{L}(\mathcal{U}),N,g\right).$$

Taking the limit $N\to\infty$ yields

$$m(Z,\alpha,\Phi,\mathcal{U},g')\leqslant e^{\varepsilon(\delta)+\gamma-\alpha}\cdot m'\left(Z,\alpha-\varepsilon(\delta)-\gamma,\Phi,\ \frac{1}{2}\mathcal{L}(\mathcal{U})\right).$$

This implies that

$$P_Z(G,\Phi,\mathcal{U},g')\leqslant P_Z'\left(G,\Phi,\ \frac{1}{2}\mathcal{L}(\mathcal{U}),g\right)+\varepsilon(\delta)+\gamma.$$

As $\delta\to 0$, that is, $\varepsilon(\delta)\to 0$, $|\mathcal{U}|\to 0$, $\gamma\to 0$, we obtain

$$P_Z(G,\Phi,g')\leqslant\liminf_{\mathcal{L}(\mathcal{U})\to 0}P_Z'\left(G,\Phi,\ \frac{1}{2}\mathcal{L}(\mathcal{U}),g\right).$$

On the other hand, fix a cover $\mathcal{U}$ of X with $|\mathcal{U}|<\delta$. Given $w\in F_m^+$, $|w|=N$ and $\mathcal{G}_w^g\subset\mathcal{S}^g(\mathcal{U})$ covering Z, we may assume without loss of generality that for every $\mathbf{U}^g\in\mathcal{G}_w^g$, we have $X_{w_{\mathbf{U}^g}}(\mathbf{U}^g)\cap Z\neq\varnothing$. Then for each

$\mathbf{U}^g \in \mathcal{G}_w^g$, we can choose $x_{\mathbf{U}^g} \in X_{w_{\mathbf{U}^g}}(\mathbf{U}^g) \cap Z$. We observe

$$X_{w_{\mathbf{U}^g}}(\mathbf{U}^g) \subset B_{w_{\mathbf{U}^g}}(2g''; x_{\mathbf{U}^g}, \delta),$$

where $g'' = g\left(\left|w_{\mathbf{U}^g}\right| + 1, |\mathcal{U}|\right)$. Denote $\mathcal{F}_w$ the collection of all the mistake Bowen balls $B_{w_{\mathbf{U}^g}}(2g''; x_{\mathbf{U}^g}, \delta)$ constructed above and then

$$M_w(Z, \alpha, \Phi, \mathcal{U}, N, g) = \inf_{\mathcal{G}_w^g}\left\{\sum_{\mathbf{U}^g \in \mathcal{G}_w^g} \exp\left(-\alpha m(\mathbf{U}) + \sup_{y \in X_{w_{\mathbf{U}^g}}(\mathbf{U}^g)} S_{w_{\mathbf{U}^g}}\Phi(y)\right)\right\}$$

$$\geqslant e^{-\alpha} \cdot \inf_{\mathcal{F}_w}\left\{\sum_{B_{w_{\mathbf{U}^g}}(2g''; x_{\mathbf{U}^g}, \delta) \in \mathcal{F}_w} \exp(-\alpha|w_{\mathbf{U}^g}| + S_{w_{\mathbf{U}^g}}\Phi(x_{\mathbf{U}^g}))\right\}$$

$$\geqslant e^{-\alpha} \cdot M'_w(Z, \alpha, \Phi, \delta, N, 2g'').$$

It follows

$$M(Z, \alpha, \Phi, \mathcal{U}, N, g) \geqslant e^{-\alpha} \cdot M'(Z, \alpha, \Phi, \delta, N, 2g'').$$

Hence

$$m(Z, \alpha, \Phi, \mathcal{U}, g) \geqslant e^{-\alpha} \cdot m'(Z, \alpha, \Phi, \delta, 2g'').$$

This implies that

$$P_Z(G, \Phi, \mathcal{U}, g) \geqslant P'_Z(G, \Phi, \delta, 2g'').$$

Taking the limit as $\delta \to 0$ gives

$$P_Z(G, \Phi, g) \geqslant \limsup_{\delta \to 0} P'_Z(G, \Phi, \delta, 2g'').$$

It is obvious that $B_w(g''; x, \delta) \subset B_w(2g''; x, \delta)$, then we obtain

$$P_Z(G, \Phi, g) \geqslant \limsup_{\delta \to 0} P'_Z(G, \Phi, \delta, g''),$$

which completes the proof.

2.3 Example

Let (X, d) be a compact metric space, $f_0, f_1, \cdots, f_{m-1}$ continuous transformations from X to itself. Similar to the definition of mean metric in [12], for any $x, y \in X$, $w \in F_m^+$, we define a mean metric $\overline{d_w}$ on X as follows:

$$\overline{d_w}(x, y) := \frac{1}{|w| + 1} \sum_{w' \leqslant \bar{w}} d(f_{w'}(x), f_{w'}(y)).$$

For $x \in X$ and $\delta > 0$, let

$$B_{\overline{d_w}}(x, \delta) := \left\{y \in X : \overline{d_w}(x, y) < \delta\right\}.$$

In 2015, Gröger and Jäger[9] gave a definition of topological entropy of the whole system in mean metric by using separated sets, and proved that the topological entropy defined by mean metric is equivalent to the topological entropy defined by Bowen metric. Similar to the process of defining topological pressure of a free semigroup G in Section 1, we can also use the center of $B_{\overline{d_w}}(x, \delta)$ to define topological pressure on non-compact subset Z, denoted by $\bar{P}_Z(G, \Phi)$.

Proposition 1 For any $Z \subset X$, we have

$$\bar{P}_Z(G, \Phi) = P_Z(G, \Phi).$$

Now, we choose $g(n, \varepsilon) = n\varepsilon$. It's clear that this function satisfies the definition of the mistake function. By the following lemma and Theorem 1, it is easy to obtain the above proposition, hence we omit the proof.

Lemma 3 For any $x \in X$, $w \in F_m^+$, and $\varepsilon > 0$, we have

$$B_w(x, \varepsilon) \subset B_{\overline{d_w}}(x, \varepsilon) \subset B_w(g; x, \sqrt{\varepsilon}).$$

Proof For any $y \in X, w \in F_m^+$, if $d_w(x,y) < \varepsilon$, then $\overline{d_w}(x,y) < \varepsilon$, so we have

$$B_w(x,\varepsilon) \subset B_{\overline{d_w}}(x,\varepsilon).$$

Set

$$I_1 = \{w' \in \mathcal{A}_w : d(f_{w'}(x), f_{w'}(y)) < \sqrt{\varepsilon}\}, \qquad I_2 = \{w' \in \mathcal{A}_w : d(f_{w'}(x), f_{w'}(y)) \geqslant \sqrt{\varepsilon}\},$$

where $\mathcal{A}_w = \{w' : w' \leqslant w\}$. Since

$$\begin{aligned}\frac{1}{|w|+1}\sum_{w' \leqslant w} d(f_{w'}(x_1), f_{w'}(x_2)) &= \frac{\sum_{w' \in I_1} d(f_{w'}(x_1), f_{w'}(x_2)) + \sum_{w' \in I_2} d(f_{w'}(x_1), f_{w'}(x_2))}{|w|+1} \\ &\geqslant \frac{\sum_{w' \in I_1} d(f_{w'}(x_1), f_{w'}(x_2)) + \sqrt{\varepsilon} \cdot \# I_2}{|w|+1} \\ &\geqslant \frac{\sqrt{\varepsilon}}{|w|+1} \cdot \# I_2,\end{aligned}$$

then if $y \in B_{\overline{d_w}}(x,\varepsilon)$, i. e. , $\overline{d_w}(x,y) < \varepsilon$, we have

$$\# I_2 \leqslant (|w|+1)\sqrt{\varepsilon}.$$

Thus, we obtain

$$\frac{\#\{w' \in \mathcal{A}_w : d(f_{w'}(x), f_{w'}(y)) \geqslant \sqrt{\varepsilon}\}}{|w|+1} \leqslant \sqrt{\varepsilon}.$$

Then $y \in B_w(g; x, \sqrt{\varepsilon})$. Therefore, we have

$$B_{\overline{d_w}}(x,\varepsilon) \subset B_w(g; x, \sqrt{\varepsilon}).$$

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（责任编辑　冯兆永）